

Bredon homology and equivariant K -homology of $SL(3, \mathbb{Z})$

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Abstract

We obtain the equivariant K -homology of the classifying space $\underline{ESL}(3, \mathbb{Z})$ from the computation of its Bredon homology with respect to finite subgroups and coefficients in the representation ring. We also obtain the corresponding results for $GL(3, \mathbb{Z})$. Our calculations give therefore the topological side of the Baum-Connes conjecture for these groups.

1 Introduction

Consider a discrete group G . The Baum-Connes conjecture [1] identifies the K -theory of the reduced C^* -algebra of G , $C_r^*(G)$, with the equivariant K -homology of a certain classifying space associated to G . This space is called the classifying space for proper actions, written \underline{EG} . The conjecture states that a particular map between these two objects, called the assembly map,

$$\mu_i : K_i^G(\underline{EG}) \longrightarrow K_i(C_r^*(G)) \quad i \geq 0,$$

is an isomorphism. Here the left hand side is the equivariant K -homology of \underline{EG} and the right hand side is the K -theory of $C_r^*(G)$. The conjecture can be stated more generally [1, Conjecture 3.15].

The equivariant K -homology and the assembly map are usually defined in terms of Kasparov's KK -theory. For a discrete group G , however, there is a more topological description due to Davis and Lück [3], and Joachim [9] in terms of spectra over the orbit category of G . We will keep in mind the topological picture of the Baum-Connes conjecture (see Mislin's notes in [12]).

Part of the importance of this conjecture is due to the fact that it is related to many other relevant conjectures in different areas of mathematics [12]. Nevertheless, the conjecture itself allows the computation of the K -theory of $C_r^*(G)$ from the K^G -homology of \underline{EG} . In turn, this K -homology can be achieved by means of the Bredon homology of \underline{EG} , as we explain later.

The Baum-Connes conjecture has been proved for some large families of groups, yet remains unsolved in general. In particular, Higson and Kasparov [7] proved the conjecture for groups having the Haagerup property (or a-T-menable), that is, groups which admits a metrically proper isometric action on

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some affine Hilbert space. On the other side, a group has Kazhdan's property T if every isometric action of G on an affine Hilbert space has a fixed point. Thus, infinite groups with the property T do not have the Haagerup property. There are not many infinite discrete groups with the property T for which the Baum-Connes conjecture has been proved (the first examples are due to Lafforgue [10]).

Consequently, the group $SL(3, \mathbb{Z})$ becomes relevant in this context since the Baum-Connes conjecture is unknown for $SL(n, \mathbb{Z})$, $n \geq 3$ and these groups have property T . On the other hand, the Baum-Connes assembly map is known to be injective for $SL(n, \mathbb{Z})$ (in general, for all closed subgroups of a Lie group with a finite number of connected components, see [1, §7]). Finally, note that there are counterexamples to the Baum-Connes conjecture for groupoids that can be constructed from $SL(3, \mathbb{Z})$, and more generally for a discrete group with property T and such that the Baum-Connes map is injective ([8, p. 338]).

In this paper we obtain the equivariant K -homology of $\underline{E}G$ for $G = SL(3, \mathbb{Z})$ and $G = GL(3, \mathbb{Z})$ from the computation of its Bredon homology. The results amount to the topological side of the Baum-Connes conjecture, $K_i^G(\underline{E}G)$. We state the results here.

Theorem 1. *The Bredon homology of $\underline{E}SL(3, \mathbb{Z})$ with coefficients in the representation ring is*

$$H_i^{\text{fin}}(\underline{E}SL(3, \mathbb{Z}); \mathcal{R}) = \begin{cases} \mathbb{Z}^{\oplus 8} & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

Corollary 2. *The equivariant K -homology of $\underline{E}G$ for $G = SL(3, \mathbb{Z})$ is*

$$K_0^G(\underline{E}G) = \mathbb{Z}^{\oplus 8}, \quad K_1^G(\underline{E}G) = 0.$$

The results for $GL(3, \mathbb{Z})$ follow from a Künneth formula for Bredon homology since $GL(3, \mathbb{Z}) = SL(3, \mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$.

Theorem 3. *The Bredon homology of $\underline{E}GL(3, \mathbb{Z})$ with coefficients in the representation ring is*

$$H_i^{\text{fin}}(\underline{E}GL(3, \mathbb{Z}); \mathcal{R}) = \begin{cases} \mathbb{Z}^{\oplus 16} & i = 0 \\ 0 & i \neq 0 \end{cases}.$$

Corollary 4. *The equivariant K -homology of $\underline{E}G$ for $G = GL(3, \mathbb{Z})$ is*

$$K_0^G(\underline{E}G) = \mathbb{Z}^{\oplus 16}, \quad K_1^G(\underline{E}G) = 0.$$

We start with a brief review on classifying spaces, Bredon homology and equivariant K -homology; then we describe a model of $\underline{E}SL(3, \mathbb{Z})$; in the last section we compute its Bredon homology and prove the theorems above. These results are part of the author's PhD thesis [14, Chapter 4].

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2 Preliminaries

2.1 Classifying space for proper actions

Let G be a discrete group. A G -CW-complex is a CW-complex with a G -action permuting the cells and such that if a cell is sent to itself, it is done by the identity map. We call the G -action *proper* if all cell stabilizers are finite subgroups of G .

Definition 1. A *model for $\underline{E}G$* is a proper G -CW-complex X such that for any proper G -CW-complex Y there is a unique G -map $Y \rightarrow X$, up to G -homotopy equivalence.

One can prove that a proper G -CW-complex X is a model of $\underline{E}G$ if and only if the subcomplex of fixed points X^H is contractible for each $H \leq G$ finite. It can be shown that classifying spaces proper actions always exists. They are clearly unique up to G -homotopy equivalence. We write \underline{BG} for the quotient $\underline{E}G/G$. Note that for *free* actions instead of proper, we recover the definition of $E\mathbb{G}$, whose quotient BG is the classifying space for principal G -bundles. See [1, §2] or [11] for details and more information on classifying spaces.

2.2 Bredon (co)homology

Given a group G and a family \mathfrak{F} of subgroups, we will write $\mathcal{O}_{\mathfrak{F}}G$ for the *orbit category*. The objects are left cosets G/K , $K \in \mathfrak{F}$, and morphisms the G -maps $\phi : G/K \rightarrow G/L$. Such a G -map is uniquely determined by its image $\phi(K) = gL$, and we have $g^{-1}Kg \subset L$. Conversely, such $g \in G$ defines a G -map.

A *left* (resp. *right*) *Bredon module* is a covariant (resp. contravariant) functor from $\mathcal{O}_{\mathfrak{F}}G$ to the category of abelian groups. Bredon modules form a category, which is abelian, and we can use homological algebra to define Bredon homology (see [12, pp. 7-10]). Nevertheless, we give now a practical definition.

Consider a G -CW-complex X , a family \mathfrak{F} of subgroups of G containing all cell stabilizers, and a left Bredon module M . The *Bredon homology groups* $H_i^{\mathfrak{F}}(X; M)$ are obtained as the homology of the following chain complex (C_*, ∂_*) . Let $\{e_\alpha\}$ be orbit representatives of the d -cells ($d \geq 0$) and write S_α for $\text{stab}(e_\alpha) \in \mathfrak{F}$. Define

$$C_d = \bigoplus_{\alpha} M(G/S_\alpha).$$

If ge' is a typical $(d-1)$ -cell in the boundary of e_α then $g^{-1}\text{stab}(e_\alpha)g \subset \text{stab}(e')$, giving a G -map (write S' for $\text{stab}(e')$)

$$\phi : G/S_\alpha \rightarrow G/S',$$

which induces a homomorphism $M(\phi) : M(G/S_\alpha) \rightarrow M(G/S')$. This yields a differential $\partial_d : C_d \rightarrow C_{d-1}$, and the Bredon homology groups $H_i^{\mathfrak{F}}(X; M)$ correspond to the homology of (C_*, ∂_*) . Bredon cohomology is defined analogously, for M a right Bredon module.

2.3 Proper actions and the representation ring

We are interested in the case $X = \underline{E}G$, $\mathfrak{F} = \mathfrak{Fin}(G)$ —the family of all finite subgroups of G — and $M = \mathcal{R}$ the complex representation ring, considered as a

Bredon module as follows. On objects we set

$$\mathcal{R}(G/K) = R_{\mathbb{C}}(K), \quad K \in \mathfrak{Fin}(G)$$

the ring of complex representations of the finite group K (viewed just as an abelian group). For a G -map $\phi : G/K \rightarrow G/L$ we have $g^{-1}Kg \subset L$ for some $g \in G$ so define $\mathcal{R}(\phi) : R_{\mathbb{C}}(K) \rightarrow R_{\mathbb{C}}(L)$ as induction from $g^{-1}Kg$ into L once we identify $R_{\mathbb{C}}(g^{-1}Kg)$ with $R_{\mathbb{C}}(K)$.

We state two useful results about the zero degree and the higher degree Bredon homology groups $H_i^{\mathfrak{Fin}}(\underline{E}G; \mathcal{R})$, and a Künneth formula.

Proposition 5. *Let G be group and denote by $\text{FC}(G)$ the set of conjugacy classes of elements of finite order in G . Then there is an isomorphism*

$$H_0^{\mathfrak{Fin}}(\underline{E}G; \mathcal{R}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}[\text{FC}(G)].$$

Proof. See Definition 3.18 and Theorem 3.19 in Mislin's notes [12]. \diamond

Consequently, the rank of the zeroth Bredon homology group above coincides with the number of conjugacy classes of elements of finite order in G .

Define the *singular set* X^{sing} of a G -set X as the subspace of the points with non-trivial stabilizers. The following is Lemma 3.21 in Mislin's notes [12].

Proposition 6. *Let G be an arbitrary group. Then there is a natural map*

$$H_i^{\mathfrak{Fin}}(\underline{E}G; \mathcal{R}) \longrightarrow H_i(\underline{B}G; \mathbb{Z}),$$

which is an isomorphism in dimensions $i > \dim(\underline{E}G^{sing}) + 1$ and injective in dimension $i = \dim(\underline{E}G^{sing}) + 1$.

There is a Künneth formula for the direct product of two groups. Given a group G , define $H_i^{\mathfrak{Fin}}(G; \mathcal{R})$ as $H_i^{\mathfrak{Fin}(G)}(X; \mathcal{R}^G)$, where X is any model of $\underline{E}G$ and \mathcal{R}^G is the representation ring as a Bredon module over $\mathcal{O}_{\mathfrak{Fin}(G)}G$ (cf. [12]).

Proposition 7. *Let G and H be two groups. For every $n \geq 0$ there is a split exact sequence*

$$\begin{aligned} 0 \rightarrow \bigoplus_{i+j=n} \left(H_i^{\mathfrak{Fin}}(G; \mathcal{R}) \otimes H_j^{\mathfrak{Fin}}(H; \mathcal{R}) \right) \rightarrow H_n^{\mathfrak{Fin}}(G \times H; \mathcal{R}) \rightarrow \\ \bigoplus_{i+j=n-1} \text{Tor}\left(H_i^{\mathfrak{Fin}}(G; \mathcal{R}), H_j^{\mathfrak{Fin}}(H; \mathcal{R})\right) \rightarrow 0. \end{aligned}$$

Proof. Details can be found in [13, §3] (see also [5]). \diamond

2.4 Equivariant K -homology

There is an equivariant version of K -homology, denoted $K_i^G(-)$ and defined in [3] (see also [9]) using spaces and spectra over the orbit category of G . It was originally defined in [1] using Kasparov's KK -theory. We will only recall the properties we need.

Equivariant K -homology satisfies Bott mod-2 periodicity, so we only consider K_0^G and K_1^G . For any subgroup H of G we have

$$K_i^G(G/H) = K_i(C_r^*(H)),$$

that is, its value at one-orbit spaces corresponds to the K -theory of the reduced C^* -algebra of the typical stabilizer. If H is a finite subgroup then $C_r^*(H) = \mathbb{C}H$ and

$$K_i^G(G/H) = K_i(\mathbb{C}H) = \begin{cases} R_{\mathbb{C}}(H) & i = 0, \\ 0 & i = 1. \end{cases}$$

This allows us to view $K_i^G(-)$ as a Bredon module over $\mathcal{O}_{\mathfrak{Fin}G}$.

We can use an equivariant Atiyah-Hirzebruch spectral sequence to compute the K^G -homology of a proper G -CW-complex X from its Bredon homology (see [12, pp. 49-50] for details), as

$$E_{p,q}^2 = H_p^{\mathfrak{Fin}}(X; K_q^G(-)) \Rightarrow K_{p+q}^G(X).$$

In the simple case when Bredon homology concentrates at low degree we deduce the following fact.

Proposition 8. *Write $H_i = H_i^{\mathfrak{Fin}}(X; \mathcal{R})$ and $K_i = K_i^G(X)$. If $H_i = 0$ for $i \geq 2$ then $K_0 = H_0$ and $K_1 = H_1$.*

3 A model for $\underline{E}SL(3, \mathbb{Z})$

3.1 The symmetric space

We describe a first model of the classifying space $\underline{E}SL(n, \mathbb{Z})$, for any $n \geq 2$ (cf. [2, pp. 38-40]). Let $Q(n)$ be the space of real, symmetric, positive definite $n \times n$ matrices (equivalently, positive definite quadratic forms on \mathbb{R}^n). Multiplication by positive scalars gives an action whose quotient $X(n) = Q(n)/\mathbb{R}^+$ is called the *symmetric space*. Since the right action of $GL(n, \mathbb{R})$ on $Q(n)$ given by $A \cdot g = g^t A g$ is transitive, with typical stabilizer $O(n)$, we can identify $Q(n)$ with $GL(n, \mathbb{R})/O(n)$. On the other hand, the inclusion of $SL(n, \mathbb{R})/SO(n)$ into $GL(n, \mathbb{R})/O(n)$ is an $SL(3, \mathbb{Z})$ -equivariant homotopy equivalence, also up to multiplication by positive scalars on the codomain. Therefore, the symmetric space $X(n)$ is $SL(3, \mathbb{Z})$ -equivariant homotopy equivalent to $SL(n, \mathbb{R})/SO(n)$.

The action of $GL(n, \mathbb{R})$ on $Q(n)$ induces an action on $X(n)$, which restricts to a $SL(n, \mathbb{Z})$ -action. As a $SL(n, \mathbb{Z})$ -space, $X(n)$ has finite stabilizers, since $SL(n, \mathbb{Z})$ is a discrete subgroup of $GL(n, \mathbb{R})$ and $O(n)$ is compact. Moreover, we have the following.

Proposition 9. *The symmetric space $X(n) \simeq SL(n, \mathbb{R})/SO(n)$ is a model of $\underline{E}SL(n, \mathbb{Z})$, of dimension $n(n+1)/2 - 1$.*

Proof. The space $Q(n)$ is clearly a (convex) cone, so contractible. Since the action is linear, the fixed point subspace $Q(n)^H$, $H \leq SL(n, \mathbb{Z})$, is also a cone. It is not empty whenever H is a finite subgroup; take for instance the ‘average point’ of the orbit of any $A \in Q(n)$,

$$\frac{1}{|H|} \sum_{h \in H} A \cdot h \in Q(n)^H.$$

Note that $A \in Q(n)^H$ if and only if its class $[A] \in X(n)^H$, so that $X(n)^H$ is neither empty if H is finite. Moreover, if we fix a representative of each class (for instance, choose a matrix norm $\|\cdot\|$ and define $A_u = A/\|A\|$), then there is a contracting homotopy $H([B], t) = [tA_u + (1-t)B_u]$ for any fixed $[A] \in X^H$. \diamond

Remark 10. More generally, if Γ is a discrete subgroup of a Lie group G with finitely many connected components, take K a maximal compact subgroup of G . Then G/K is a model for $\underline{E}G$ and, therefore, a model for $\underline{E}\Gamma$ [1].

3.2 Deformation retractions

There is a better model of $\underline{E}SL(n, \mathbb{Z})$, obtained as an $SL(n, \mathbb{Z})$ -equivariant deformation retract of the symmetric space $X(n)$, of dimension $n(n-1)/2$. It is obtained via reduction theory of quadratic forms (see [2, pp. 213-17]). However, finding an explicit cellular decomposition and describing the stabilizers is in general quite laborious. For $SL(3, \mathbb{Z})$ this has been done by Soulé in [15] and also by Henn in [6]. We now describe (without proofs) the deformation retract and orbit space for $n = 3$, following Soulé.

Denote the elements of $Q = Q(3)$, respectively of $X = X(3)$, as

$$A = (a_{ij}) \in Q, \quad [A] = \{\lambda A \mid \lambda \in \mathbb{R}^+\} \in X.$$

Recall the right action of $SL(3, \mathbb{Z})$ on Q , $A \cdot g = g^t A g$, which extends to an action on X . Each orbit in X has a total preorder as follows. Given $A = B \cdot g$ we say that $[A] < [B]$ if the sequence of diagonal elements of A is smaller than the one of B with respect to the lexicographic order in \mathbb{R}^3 . (This is well-defined : if $\lambda A = B \cdot g'$ then $\lambda B = B \cdot (g'g^{-1})$ so $\lambda = 1$ and $g = g'$.) An element $[A] \in X$ is called *reduced* if it is minimal in its orbit. Define the subspace

$$Y = \{[A] \text{ reduced and } a_{11} = a_{22} = a_{33}\}.$$

Proposition 11. *The space Y is an $SL(3, \mathbb{Z})$ -deformation retract of X and therefore a model of $\underline{E}SL(3, \mathbb{Z})$, of dimension 3.*

Proof. The result follows from Theorem 1 in [15]. \diamond

Remark 12. The minimal dimension for a model of $\underline{E}SL(3, \mathbb{Z})$ is actually three. In general, one can prove that the strict upper triangular group in $SL(n, \mathbb{Z})$ has cohomological dimension $n(n-1)/2$, so any model of $\underline{E}SL(n, \mathbb{Z})$ has dimension at least that; see [2, Chp. VIII].

3.3 Description of the orbit space

In this section we give an equivariant cellular decomposition and describe the stabilizers for the orbit space $SL(3, \mathbb{Z}) \backslash Y = \underline{B}SL(3, \mathbb{Z})$. We follow the approach and notation in Soulé's paper [15], although an equivalent and detailed description of this model can be found in Henn's [6].

Let C be the truncated cube of \mathbb{R}^3 with centre $(0, 0, 0)$ and side length 2, truncated at the vertices $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, -1)$, through the mid-points of the corresponding sides (Figure 1). An element in Y can be uniquely written as the class of a matrix

$$A = \begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix} = A(x, y, z),$$

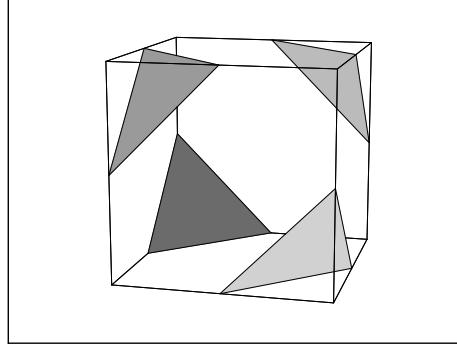


Figure 1: Truncated cube C

so that we can identify $[A]$ with the point $(x, y, z) \in \mathbb{R}^3$. It can be shown that $A(x, y, z)$ is reduced if and only if $(x, y, z) \in C$. Let E be the subspace of C given by the points (x, y, z) satisfying

$$\begin{aligned} |z| &\leq y \leq x \leq 1 \\ z - x - y + 2 &\geq 0. \end{aligned}$$

We can give an explicit triangulation of E as shown in Figure 2. The vertices are

$$\begin{aligned} O &= (0, 0, 0) & Q &= (1, 0, 0) \\ M &= (1, 1, 1) & N &= (1, 1, 1/2) \\ M' &= (1, 1, 0) & N' &= (1, 1/2, -1/2) \\ P &= (2/3, 2/3, -2/3). \end{aligned}$$

Note that the elements of $SL(3, \mathbb{Z})$

$$q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } q_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

send M, N, Q to M', N', Q and N, N', M', Q to N', N, M', Q respectively. Consequently, we must identify the following simplices in the quotient

$$\begin{aligned} M &\equiv M', \quad N \equiv N', \quad QM \equiv QM', \quad QN \equiv QN', \quad MN \equiv M'N \equiv M'N' \\ \text{and } QMN &\equiv QM'N \equiv QM'N'. \end{aligned}$$

This identifications correspond to folding over the triangles QMN , QNM' and $QM'N'$ along the edges QN and QM' respectively.

Theorem 13. *The space E with the identifications above is a 3-dimensional model of $\underline{B}SL(3, \mathbb{Z})$.*

Proof. See Theorem 2 in [15] or Theorem 2.4 in [6]. ◇

We now describe the orbits of cells and corresponding stabilizers. This can be found in Theorem 2 of Soulé's article (although we use a cellular structure instead of a simplicial one) or in Section 2.5 of Henn's work. We have changed

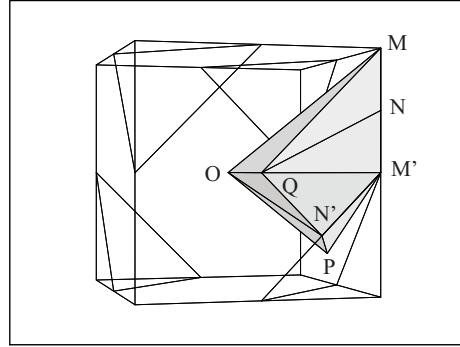


Figure 2: Triangulation of the fundamental domain E

the chosen generators so that they agree with the presentations on page 10. We summarize the information in the following table.

Vertices				2-cells			
v_1	O	g_2, g_3	S_4	t_1	OQM	g_2	C_2
v_2	Q	g_4, g_5	D_6	t_2	$QM'N$	g_1	$\{1\}$
v_3	M	g_6, g_7	S_4	t_3	$M'N'P$	g_{12}, g_{14}	$C_2 \times C_2$
v_4	N	g_6, g_8	D_4	t_4	$OQN'P$	g_5	C_2
v_5	P	g_5, g_9	S_4	t_5	$OMM'P$	g_6	C_2
Edges				3-cell			
e_1	OQ	g_2, g_5	$C_2 \times C_2$	T_1		g_1	$\{1\}$
e_2	OM	g_6, g_{10}	D_3				
e_3	OP	g_6, g_5	D_3				
e_4	QM	g_2	C_2				
e_5	QN'	g_5	C_2				
e_6	MN	g_6, g_{11}	$C_2 \times C_2$				
e_7	$M'P$	g_6, g_{12}	D_4				
e_8	$N'P$	g_5, g_{13}	D_4				

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. We use the following notation: $\{1\}$ denotes the trivial group, C_n the cyclic group of n elements, D_n the dihedral group with $2n$ elements and S_n the symmetric group of permutations on n objects. The generating

elements referred to above are

$$\begin{aligned}
g_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & g_2 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} & g_3 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\
g_4 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} & g_5 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & g_6 &= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\
g_7 &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} & g_8 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} & g_9 &= \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\
g_{10} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} & g_{11} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} & g_{12} &= \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
g_{13} &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} & g_{14} &= \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.
\end{aligned}$$

Finally, the cell boundaries can be easily read from Figure 2, once we fix an orientation; namely, the ordering of the vertices $O < Q < M < M' < N < N' < P$ induces an orientation in E and also in $\underline{BSL}(3, \mathbb{Z}) = E/\sim$. Thus, the boundaries with respect to the orbit representatives are (recall the identifications given by q_1 and q_2)

$$\begin{aligned}
\partial e_1 &= v_2 - v_1 & \partial e_2 &= v_3 - v_1 & \partial e_3 &= v_5 - v_1 & \partial e_4 &= v_3 - v_2 \\
\partial e_5 &= v_4 \cdot q_2 - v_2 & \partial e_6 &= v_4 - v_3 & \partial e_7 &= v_5 - v_3 \cdot q_1 & \partial e_8 &= v_5 - v_4 \cdot q_2 \\
\partial t_1 &= e_1 - e_2 + e_4 & \partial t_2 &= e_4 \cdot q_1 - e_5 \cdot q_2 + e_6 \cdot q_1 q_2 \\
\partial t_3 &= e_6 \cdot q_1 - e_7 + e_8 & \partial t_4 &= e_1 - e_3 + e_5 + e_8 \\
\partial t_5 &= e_2 - e_3 + e_6 - e_6 \cdot q_1 q_2 + e_7 & \partial T_1 &= -t_1 + t_2 - t_3 + t_4 - t_5.
\end{aligned}$$

We now have all the information needed to compute the corresponding Bredon homology, which is carried out in the next section.

4 Bredon homology of $\underline{ESL}(3, \mathbb{Z})$

The Bredon chain complex associated to $Y = \underline{ESL}(3, \mathbb{Z})$ is

$$\begin{aligned}
0 \longrightarrow R_{\mathbb{C}}(\text{stab}(T_1)) &\xrightarrow{\Psi_3} \bigoplus_{i=1}^5 R_{\mathbb{C}}(\text{stab}(t_i)) \xrightarrow{\Psi_2} \\
&\bigoplus_{j=1}^8 R_{\mathbb{C}}(\text{stab}(e_j)) \xrightarrow{\Psi_1} \bigoplus_{k=1}^5 R_{\mathbb{C}}(\text{stab}(v_k)) \longrightarrow 0,
\end{aligned} \tag{4.1}$$

where the Ψ_i 's are given by induction among representation rings: if the boundary of a d -cell e^d is, in terms of orbit representatives of $(d-1)$ -cells,

$$\partial e^d = \sum_{j=1}^n e_j^{d-1} \cdot g_j$$

and $\tau \in R_{\mathbb{C}}(\text{stab}(e^d))$, then $\Psi_d(\tau) = \tau \uparrow \text{stab}(e_j^{d-1})$, where \uparrow represents induction with respect to $g \cdot \text{stab}(e^d) \cdot g^{-1} \subset \text{stab}(e_j^{d-1})$. Note that we write $\rho \uparrow H$ for induction into a supergroup H and $\rho \downarrow H$ for restriction into a subgroup (we omit the group when it is clear from the context).

To compute $\Psi_d(\tau)$, we use two basic facts:

- any representation (or character) can be uniquely written as a sum of irreducible ones $\tau = n_1\rho_1 + \dots + n_s\rho_s$, with $n_i = (\tau|\rho_i)$ and $(\cdot|\cdot)$ the usual scalar product of characters;
- Frobenius reciprocity: $(\tau \uparrow |\rho)_H = (\tau|\rho \downarrow)_{H'}$ where $H' \leq H$.

Firstly, we note down the character tables of the groups appearing as cell stabilizers. We write $\langle g \rangle$ for the group generated by an element g , and $\langle \text{gens} \mid \text{rels} \rangle$ for a presentation of a group.

Trivial group $\{1\} = \langle g_1 \rangle$

$\{1\}$	g_1
τ	1

Cyclic group $C_2 = \langle g_i \rangle$

C_2	1	g_i
ρ_1	1	1
ρ_2	1	-1

Dihedral group $D_n = \langle g_i, g_j \rangle = \langle g_i, g_j \mid (g_i)^2 = (g_j)^2 = (g_i g_j)^n = 1 \rangle$

D_n	$(g_i g_j)^k$	$g_j (g_i g_j)^k$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi_3}$	$(-1)^k$	$(-1)^k$
$\widehat{\chi_4}$	$(-1)^k$	$(-1)^{k+1}$
ϕ_p	$2 \cos(2\pi p k/n)$	0

where $0 \leq k \leq n-1$, p varies from 1 to $n/2-1$ (n even) or $(n-1)/2$ (n odd) and the hat $\widehat{}$ denotes a character which only appears when n is even. Note that $C_2 \times C_2 \cong D_2$.

Symmetric group $S_4 = \langle g_i, g_j \rangle$ where g_i is a transposition and g_j a cycle of length 4. The character table, in cycle type notation, is

S_4	1	(12)	(123)	(1234)	(12)(34)
π_1	1	1	1	1	1
π_2	1	-1	1	-1	1
π_3	2	0	-1	0	2
π_4	3	1	0	-1	-1
π_5	3	-1	0	1	-1

In the next sections we will need to agree on an ordering of the conjugacy classes for each cell stabilizer. The same applies for the irreducible characters. We fix

both orderings as shown in the character tables above. Note that for a dihedral group the arranging of conjugacy classes (in terms of representatives) is

$$\begin{aligned} n \text{ odd: } & 1, g_i g_j, (g_i g_j)^2, \dots, (g_i g_j)^{(n-1)/2}, g_j \\ n \text{ even: } & 1, g_i g_j, (g_i g_j)^2, \dots, (g_i g_j)^{n/2}, g_j, g_j (g_i g_j). \end{aligned}$$

Considering the ranks of the corresponding representation rings, that is, the number of irreducible characters, we can view the Bredon chain complex (4.1) as (write $n \cdot \mathbb{Z}$ for $\mathbb{Z}^{\oplus n}$)

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Psi_3} 11 \cdot \mathbb{Z} \xrightarrow{\Psi_2} 28 \cdot \mathbb{Z} \xrightarrow{\Psi_1} 26 \cdot \mathbb{Z} \longrightarrow 0. \quad (4.2)$$

4.1 Computation of Ψ_3

Denote by ∂e the boundary of a d -cell e in terms of $(d-1)$ -cells. Let τ be the trivial representation of $\text{stab}(T_1) = \{1\}$. We have

$$\partial T_1 = \sum_{i=1}^5 (-1)^i t_i \Rightarrow \Psi_3(\tau) = \sum_{i=1}^5 (-1)^i \tau \uparrow \text{stab}(t_i).$$

Inducing the trivial representation gives the regular representation $\rho_1 + \dots + \rho_s$ so Ψ_3 is, written as a matrix of an homomorphism of free abelian groups,

$$\Psi_3(1) = \left(\begin{array}{cc|cc|cc|cc|cc} -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 \end{array} \right).$$

Here each group of ± 1 s indicates the elements corresponding to each representation ring $R_{\mathbb{C}}(\text{stab}(t_i))$. This matrix reduces by elementary operations to

$$\Psi_3 \equiv (1 \ 0 \ \dots \ 0).$$

In particular, $H_3 = \ker \Psi_3 = 0$ and $\text{im } \Psi_3 \cong \mathbb{Z}$.

4.2 Computation of Ψ_2

For each 2-cell, we work out the induction map for each inclusion (possibly after conjugation) of stabilizers. For t_1 , we have $\partial t_1 = e_1 - e_2 + e_4$ so $\text{stab}(t_1) \subset \text{stab}(e_i)$ for $i = 1, 2, 4$. The first inclusion is

$$C_2 = \text{stab}(t_1) = \langle g_2 \rangle \subset \langle g_2, g_5 \rangle = \text{stab}(e_1) = C_2 \times C_2 = D_2.$$

Consider the irreducible characters ρ_1, ρ_2 in $R_{\mathbb{C}}(\text{stab}(t_1))$ and χ_1, \dots, χ_4 in $R_{\mathbb{C}}(\text{stab}(e_1))$ as on the character tables on page 10. Then

	1	g_2	$(\rho_1 \mid \chi_j \downarrow)$	$(\rho_2 \mid \chi_j \downarrow)$
$\chi_1 \downarrow$	1	1	1	0
$\chi_2 \downarrow$	1	-1	0	1
$\chi_3 \downarrow$	1	-1	0	1
$\chi_4 \downarrow$	1	1	1	0

Therefore,

$$\begin{aligned}\rho_1 \uparrow &= \chi_1 + \chi_4 \\ \rho_2 \uparrow &= \chi_2 + \chi_3.\end{aligned}$$

The corresponding submatrix representing $R_{\mathbb{C}}(\text{stab}(t_1)) \xrightarrow{\text{Ind}} R_{\mathbb{C}}(\text{stab}(e_1))$ is

$$\begin{array}{c|cccc} & \chi_1 & \chi_2 & \chi_3 & \chi_4 \\ \rho_1 \uparrow & 1 & 0 & 0 & 1 \\ \rho_2 \uparrow & 0 & 1 & 1 & 0 \end{array}$$

For the inclusion of $\text{stab}(t_1) = \langle g_2 \rangle$ into $\text{stab}(e_2) = \langle g_6, g_{10} \rangle$, we have $g_2 = g_{10}g_6g_{10}$, and

$$\begin{array}{c|cc|cc} & 1 & g_{10}(g_6g_{10}) & (\rho_1 \mid \chi_j \downarrow) & (\rho_2 \mid \chi_j \downarrow) \\ \chi_1 \downarrow & 1 & 1 & 1 & 0 \\ \chi_2 \downarrow & 1 & -1 & 0 & 1 \\ \phi_1 \downarrow & 2 & 0 & 1 & 1 \end{array}$$

Consequently,

$$\begin{aligned}\rho_1 \uparrow &= \chi_1 + \phi_1 \\ \rho_2 \uparrow &= \chi_3 + \phi_1.\end{aligned}$$

The corresponding submatrix is (note the minus sign, since the cell appears as $-e_2$ in ∂t_1)

$$\begin{array}{c|cccc} & & & e_2 & \\ t_1 & -1 & 0 & -1 & \\ & 0 & -1 & -1 & \end{array}$$

Here we have simplified the notation; a matrix with top label e and left label t will represent the coefficients of the induced characters of $R_{\mathbb{C}}(\text{stab}(t))$ into $R_{\mathbb{C}}(\text{stab}(e))$, possibly after conjugation.

Now, $\text{stab}(t_1) = \text{stab}(e_4)$, so induction gives the identity map. In brief:

$$\begin{array}{c|ccc|ccc} & e_1 & & & e_2 & & & e_4 \\ t_1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ & 0 & 1 & 1 & 0 & 0 & -1 & -1 \end{array}$$

The process is analogous for the other 2-cells. All the inclusions among stabilizers are immediate except the ones listed below.

$$\begin{aligned}\text{stab}(t_3) &\subset \text{stab}(e_6 \cdot q_1), & \text{stab}(e_7), & \text{stab}(e_8) \\ g_{12} &= q_1^{-1}g_6q_1 & = g_{12} & = g_{13}(g_5g_{13})^2 \\ g_{14} &= q_1^{-1}g_{11}q_1 & = g_{12}(g_6g_{12})^2 & = (g_5g_{13})^2. \\ \text{stab}(t_5) &\subset \text{stab}(e_6 \cdot q_1q_2) \\ g_6 &= (q_1q_2)^{-1}g_6g_{11}(q_1q_2).\end{aligned}$$

We now show the results, which can be easily verified by a careful reader.

$$\begin{array}{c|ccc|ccc} & e_4 & & & e_5 & & & e_6 \\ t_2 & 1 & 1 & | & -1 & -1 & | & 1 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{c}
\begin{array}{c} e_6 & e_7 & e_8 \end{array} \\
\begin{array}{c} t_3 \left| \begin{array}{cccc|cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right| \end{array} \\
\begin{array}{c} e_1 & e_3 & e_5 & e_8 \end{array} \\
\begin{array}{c} t_4 \left| \begin{array}{cccc|cccc|cccc} 1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right| \end{array} \\
\begin{array}{c} e_2 & e_3 & e_6 & e_7 \end{array} \\
\begin{array}{c} t_5 \left| \begin{array}{ccc|ccc|ccc} 1 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right| \end{array}
\end{array}$$

The submatrices above amount to an 11×28 matrix representing Ψ_2 . This matrix can be reduced to its normal form consisting of the identity of size 10 and zeroes elsewhere,

$$\Psi_2 \equiv \left(\begin{array}{c|c} Id_{10} & 0 \\ \hline 0 & 0 \end{array} \right).$$

4.3 Computation of Ψ_1

The computations for Ψ_1 are similar and straightforward. The relevant inclusions among stabilizers are the following. We give a conjugacy representative as $(\sim g_i)$ when necessary.

$$\begin{array}{llll}
\text{stab}(e_1) & \subset \text{stab}(v_1), & \text{stab}(v_2) & \\
g_2 & = g_2 & = g_5(g_4g_5)^3 & \\
g_5 & = g_3g_2g_3^{-1}g_2g_3 (\sim g_2) & & \\
\text{stab}(e_2) & \subset \text{stab}(v_1), & \text{stab}(v_3) & \\
g_6 & = g_3g_2g_3^{-1} & = & \\
g_{10} & = g_2g_3g_2g_3^{-1}g_2 (\sim g_2) & = & g_7^{-1}g_6g_7 \\
\text{stab}(e_3) & \subset \text{stab}(v_1), & \text{stab}(v_5) & \\
g_6 & = g_3g_2g_3^{-1} & = & g_9^{-1}g_5g_9 \\
g_5 & = g_3g_2g_3^{-1}g_2g_3 (\sim g_2) & & \\
\text{stab}(e_4) & \subset \text{stab}(v_2), & \text{stab}(v_3) & \\
g_6 & = g_5(g_4g_5)^3 & = & g_6g_7^2g_6g_7^{-1} (\sim g_6) \\
\text{stab}(e_5) & \subset \text{stab}(v_4 \cdot q_2) & & \\
g_5 & = q_2^{-1}g_8q_2 & & \\
\text{stab}(e_6) & \subset \text{stab}(v_3), & \text{stab}(v_4) & \\
g_{11} & = g_7g_6g_7^{-1}g_6g_7 (\sim g_6) & = & (g_6g_8)^2 \\
\text{stab}(e_7) & \subset \text{stab}(v_3 \cdot q_1), & \text{stab}(v_5) & \\
g_6 & = q_1^{-1}(g_6g_7^2g_6)q_1 (\sim g_6) & = & g_9^{-1}g_5g_9 \\
g_{12} & = q_1^{-1}g_6q_1 & = & g_9^2 (\sim g_5) \\
\text{stab}(e_8) & \subset \text{stab}(v_4 \cdot q_2), & \text{stab}(v_5) & \\
g_5 & = q_2^{-1}g_8q_2 & & \\
g_{13} & = q_2^{-1}g_6q_2 & = & g_5g_9^2g_5 (\sim g_5).
\end{array}$$

The matrices representing induction among stabilizers are the following.

$$\begin{array}{c}
e_1 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_1 & & & & & v_2 & & & & & \\ \hline -1 & 0 & -1 & -1 & 0 & & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & -1 & & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & & 0 & 0 & 0 & 1 & 1 & 0 \end{array} \right. \\ \\[10pt]
e_2 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_1 & & & & & v_3 & & & & & \\ \hline -1 & 0 & 0 & -1 & 0 & & 1 & 0 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 0 & -1 & & 0 & 1 & 0 & 0 & 1 & \\ 0 & 0 & -1 & -1 & -1 & & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right. \\ \\[10pt]
e_3 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_1 & & & & & v_5 & & & & & \\ \hline -1 & 0 & 0 & -1 & 0 & & 1 & 0 & 0 & 1 & 0 & \\ 0 & -1 & 0 & 0 & -1 & & 0 & 1 & 0 & 0 & 1 & \\ 0 & 0 & -1 & -1 & -1 & & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right. \\ \\[10pt]
e_4 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_2 & & & & & v_3 & & & & & \\ \hline -1 & 0 & 0 & -1 & -1 & -1 & & 1 & 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 & -1 & -1 & & 0 & 1 & 1 & 1 & 2 \end{array} \right. \\ \\[10pt]
e_5 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_2 & & & & & v_4 & & & & & \\ \hline -1 & 0 & -1 & 0 & -1 & -1 & & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & -1 & & 0 & 1 & 0 & 1 & 1 \end{array} \right. \\ \\[10pt]
e_6 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_3 & & & & & v_4 & & & & & \\ \hline -1 & 0 & -1 & -1 & 0 & & 1 & 0 & 0 & 1 & 0 & \\ 0 & -1 & -1 & 0 & -1 & & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & -1 & -1 & & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & -1 & -1 & & 0 & 0 & 0 & 0 & 1 & \end{array} \right. \\ \\[10pt]
e_7 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_3 & & & & & v_5 & & & & & \\ \hline -1 & 0 & -1 & 0 & 0 & & 1 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & -1 & 0 & & 0 & 1 & 1 & 0 & 0 & \\ 0 & -1 & -1 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & -1 & -1 & & 0 & 0 & 0 & 1 & 1 & \end{array} \right. \\ \\[10pt]
e_8 \left| \begin{array}{cccccc|cccccc} & & & & & & & & & & & \\ & v_4 & & & & & v_5 & & & & & \\ \hline -1 & 0 & 0 & 0 & 0 & & 1 & 0 & 1 & 0 & 0 & \\ 0 & -1 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & -1 & 0 & & 0 & 1 & 1 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0 & & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & -1 & & 0 & 0 & 0 & 1 & 1 & \end{array} \right. \end{array}$$

Altogether they form a 28×26 matrix whose normal form is

$$\left(\begin{array}{c|c} Id_{18} & 0 \\ \hline 0 & 0 \end{array} \right).$$

Remark 14. The computations have been verified with the help of a computer. In fact, the author has implemented a program in GAP [4], which computes the Bredon homology with coefficients in the representation ring of any finite proper G -CW-complex, from the cell stabilizers and boundaries. Details and the code for the algorithm can be found in [14, Appendix A].

We can now determine the Bredon homology of $SL(3, \mathbb{Z})$ from the chain maps above. Note that if we have an exact sequence of free abelian groups

$$\cdots \longrightarrow n \cdot \mathbb{Z} \xrightarrow{f} m \cdot \mathbb{Z} \xrightarrow{g} k \cdot \mathbb{Z} \longrightarrow \cdots$$

with f and g represented by matrices A and B for some fixed basis, then the homology at $m \cdot \mathbb{Z}$ is

$$\ker(g) / \text{im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \mathbb{Z}/d_s\mathbb{Z} \oplus (m - s - r) \cdot \mathbb{Z},$$

where $r = \text{rank}(B)$ and d_1, \dots, d_s are the elementary divisors of A . In our case, we obtain that the Bredon homology of $\underline{E}SL(3, \mathbb{Z})$ is

$$\begin{aligned} H_0^{\mathfrak{Fin}}(\underline{E}\Gamma; \mathcal{R}) &\cong \mathbb{Z}^{\oplus 8} \\ H_i^{\mathfrak{Fin}}(\underline{E}\Gamma; \mathcal{R}) &= 0 \quad \forall i \neq 0. \end{aligned}$$

Remark 15. These results agree with the Bredon homology expected at degree 0 and 3. The dimension of the singular part of our model of $\underline{E}\Gamma$ is 2, and $\underline{B}\Gamma$ is contractible, so we have $H_i = 0$ for all $i \geq 3$, by Proposition 6. On the other hand, $\text{rank}(H_0) = 8$ must be the number of conjugacy classes of elements of finite order of $SL(3, \mathbb{Z})$, by Proposition 5; the latter can be deduced, for instance, from the list of all finite subgroups (up to conjugacy) of $SL(3, \mathbb{Z})$ in [16]. Also, 8 is the alternating sum of the ranks in the Bredon chain complex (4.2).

4.4 Equivariant K -homology

Since the Bredon homology concentrates at degree 0, it coincides with the K^G -homology (Proposition 8), that is,

$$\begin{aligned} K_0^G(\underline{E}\Gamma) &= \mathbb{Z}^{\oplus 8} \\ K_1^G(\underline{E}\Gamma) &= 0. \end{aligned}$$

This amounts to the topological side of the Baum-Connes conjecture and injects into the analytical side, that is, the K -theory of $C_r^*(SL(3, \mathbb{Z}))$.

4.5 Results for $GL(3, \mathbb{Z})$

We have the direct product decomposition $GL(3, \mathbb{Z}) = SL(3, \mathbb{Z}) \times C_2$. We can therefore use the Künneth formula for Bredon homology from Proposition 7. Since C_2 is finite, a one-point space is a model for $\underline{E}C_2$ and its Bredon homology is $R_{\mathbb{C}}(C_2) \cong \mathbb{Z}^{\oplus 2}$ at degree 0 and vanishes elsewhere. Consequently,

$$\begin{aligned} H_0^{\mathfrak{Fin}}(\underline{E}GL(3, \mathbb{Z}); \mathcal{R}) &= H_0^{\mathfrak{Fin}}(\underline{E}SL(3, \mathbb{Z}); \mathcal{R}) \otimes H_0^{\mathfrak{Fin}}(\underline{E}C_2; \mathcal{R}) \cong \mathbb{Z}^{\oplus 16} \\ H_i^{\mathfrak{Fin}}(\underline{E}GL(3, \mathbb{Z}); \mathcal{R}) &= 0, \quad \text{if } i \neq 0. \end{aligned}$$

As before, these groups coincide with the equivariant K -homology:

$$\begin{aligned} K_0^G(\underline{E}GL(3, \mathbb{Z})) &\cong \mathbb{Z}^{\oplus 16} \\ K_1^G(\underline{E}GL(3, \mathbb{Z})) &= 0. \end{aligned}$$

References

- [1] P. Baum, A. Connes, and N. Higson, *Classifying spaces for proper actions and K-theory of group C^* -algebras*, Contemporary Mathematics **167** (1994), 241–291.
- [2] K. S. Brown, *Cohomology of Groups*, Springer Graduate Text in Mathematics, no. 87, Springer-Verlag, 1982.
- [3] J. F. Davis and W. Lück, *Spaces over a Category and Assembly Maps in Isomorphism Conjectures in K- and L-Theory*, K-Theory **15** (1998), 201–252.
- [4] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2004, (<http://www.gap-system.org>).
- [5] M. Goliński and D. L. Gonçalves, *Generalized Eilenberg-Zilber type theorem and its equivariant applications*, Bull. Sci. Math. **123** (1999), no. 4, 285–298.
- [6] H.-W. Henn, *The cohomology of $SL(3, \mathbf{Z}[1/2])$* , K-Theory **16** (1999), no. 4, 299–359.
- [7] N. Higson and G. Kasparov, *Operator K-theory for groups which act properly and isometrically on Hilbert space*, Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 131–142 (electronic).
- [8] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes Conjecture*, Geom. Funct. Anal. **12** (2002), no. 2, 330–354.
- [9] M. Joachim, *K-homology of C^* -categories and symmetric spectra representing K-homology*, Math. Ann. **327** (2003), no. 4, 641–670.
- [10] V. Lafforgue, *Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p -adique et pour certains groupes discrets possédant la propriété (T)*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 5, 439–444.
- [11] I. J. Leary and B. E. A. Nucinkis, *Every CW-complex is a classifying space for proper bundles*, Topology **40** (2001), 539–550.
- [12] G. Mislin and A. Valette, *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, 2003.
- [13] R. Sánchez-García, *Equivariant K-homology for some Coxeter groups*, arXiv:math.KT/0604402.
- [14] ———, *Equivariant K-homology of the classifying space for proper actions*, Ph.D. thesis, University of Southampton, 2005.
- [15] C. Soulé, *The cohomology of $SL_3(\mathbf{Z})$* , Topology **17** (1978), 1–22.
- [16] K.-i. Tahara, *On the finite subgroups of $GL(3, \mathbf{Z})$* , Nagoya Math. J. **41** (1971), 169–209.